

Research Article

Bifurcations and Periodic Solutions for an Algae-Fish Semicontinuous System

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We propose an algae-fish semicontinuous system for the Zeya Reservoir to study the control of algae, including biological and chemical controls. The bifurcation and periodic solutions of the system were studied using a Poincaré map and a geometric method. The existence of order-1 periodic solution of the system is discussed. Based on previous analysis, we investigated the change in the location of the order-1 periodic solution with variable parameters and we described the transcritical bifurcation of the system. Finally, we provided a series of numerical results to illustrate the feasibility of the theoretical results. These results may help to facilitate a better understanding of algal control in the Zeya Reservoir.

1. Introduction

The economic development of human society means that the waters of lakes, marshes, and reservoirs are experiencing increasingly serious eutrophication, which can cause sustained algal growth. With a high level of eutrophication, algae with rapid growth characteristics may form algal blooms, which can lead to ecological failure and even cause harm to humans. For example, algal blooms due to eutrophication appear frequently in the Zeya Reservoir in Wenzhou, which is located in a subtropical region, and this may cause deterioration in the water quality that could deprive millions of people of drinking water.

Therefore, it is necessary to control algal growth. Indeed, many researchers have studied these ecological systems, including the use of biological and chemical controls, and these systems have been described using impulsive differential equations. The theory of impulsive differential equations has experienced a period of intensive development [1–3]. These studies are concerned mainly with the properties of their solutions, such as existence, uniqueness, stability, boundedness, and periodicity, as well as the potential applications of these theories in ecosystems. In applied studies,

most investigations using impulsive differential equations have focused on systems where the impulses have fixed times [4–8].

In many practical cases, however, such as algal blooms and pest control, the impulses often depend on the state rather than fixed time periods. Thus, semicontinuous dynamic systems have been introduced for these purposes. In this study, the so-called semicontinuous dynamic system is defined using a set of impulsive state-dependent differential equations [9, 10], where the solutions are piecewise continuous functions [11]. The application of semicontinuous dynamic systems to ecosystems has been studied in the last decade [12–15]. In particular, in the literature [14], the authors find chaos because of impulsive effect. It is well known that chaos is very important for dynamical studies. A lot of scientific workers are attracted by chaotic investigation. For example, in the literature [16], Bianca and Rondoni studied a chaotic model with flat obstacles. In their work, analytical and numerical investigations support the idea that this model of transport of matter has both chaotic and nonchaotic steady states with a quite peculiar sensitive dependence on the field and on the geometry, not observed before [16]. These results which they got are very important for studies of chaos.

In this paper, we consider a semicontinuous ecological system. The main difference between our results and those described in [12–15] is that we discuss the change in location of the order-1 periodic solution with variable parameters. The system is described as follows:

$$\begin{aligned} \frac{dA}{dt} &= rA \left(1 - \frac{A}{K}\right) - \frac{aAF}{F + adA}, \\ \frac{dF}{dt} &= \frac{\varepsilon aAF}{F + adA} - mF, \quad A < h, \\ \Delta A &= -pA, \quad \Delta F = qF + \tau, \quad A = h, \end{aligned} \quad (1)$$

where A denotes the algae population density, F denotes the fish population density, r is the intrinsic per capita algae population growth rate, a is the grazing rate of fish on algae, ε is the prey assimilation efficiency of fish, K is the carrying capacity, d is the handling time, and m is the mortality and respiration rate of fish. The parameters $p \in (0, 1)$, $h > 0$, $\tau \geq 0$, and $q > -1$ represent fishes being harvested when $q \in (-1, 0)$ and released when $q \in (0, +\infty)$, $\Delta A(t) = A(t^+) - A(t)$, $\Delta F(t) = F(t^+) - F(t)$.

This paper is organized as follows. Section 2 provides some background information. Section 3 discusses the existence of an order-1 periodic solution, the change in the location of the order-1 periodic solution with variable parameters, and the transcritical bifurcation. Section 4 provides numerical results for the theory we present while the conclusions are stated in the final section.

2. Preliminaries

We consider an autonomous system with an impulse effect as

$$\begin{aligned} \frac{dx}{dt} &= P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (x, y) \notin M, \\ \Delta x &= f(x, y), \quad \Delta y = g(x, y), \quad (x, y) \in M, \end{aligned} \quad (2)$$

where $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$, and $P, Q, f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$, $M \subset \mathbb{R}^2$ are the set of impulses. It is assumed that P, Q, f , and g are all continuous with respect to x, y in \mathbb{R}^2 so the points in $M \subset \mathbb{R}^2$ lie on a line. For each point $S(x, y) \in M$, $I : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined as

$$\begin{aligned} I(S) &= S^+ = (x^+, y^+) \in \mathbb{R}^2, \\ x^+ &= x + f(x, y), \quad y^+ = y + g(x, y). \end{aligned} \quad (3)$$

Let $N = I(M)$ be the phase set of M , where $N \cap M = \emptyset$. System (2) is generally known as a semicontinuous dynamic system.

Definition 1 (see [11]). Let Γ be a first-order periodic solution of system (2), and we say that Γ is

- (1) orbitally stable if for all $\varepsilon > 0$, $\exists p \in N$, $p \in \Gamma$, and $\exists \delta > 0$ such that for all $p_1 \in \cup(p, \delta)$, $\rho(\pi(p_1, t), \Gamma) < \varepsilon$ when $t > t_0$;

- (2) orbitally semistable if for all $\varepsilon > 0$, $\exists p \in N$, $p \in \Gamma$, and $\exists \delta > 0$ such that for all $p_1 \in (p, p + \delta)$ (or $(p - \delta, p)$), $\rho(\pi(p_1, t), \Gamma) < \varepsilon$ when $t > t_0$;
- (3) orbitally attractive if for all $\varepsilon > 0$ and for all $p_2 \in N$, $\exists T > 0$ such that $\rho(\pi(p_2, t), \Gamma) < \varepsilon$ when $t > T + t_0$;
- (4) orbitally asymptotically stable if it is orbitally stable and orbitally attractive.

In this discussion, $\cup(p, \delta)$ denotes a δ -neighborhood of the point $p \in N$, $\rho(\pi(p_1, t), \Gamma)$ is the distance from $\pi(p_1, t)$ to Γ , and $\pi(p_1, t)$ is the solution of system (2) that satisfies the initial condition $\pi(p_1, t_0) = p_1$.

Definition 2. The phase plane is divided into two parts by the trajectory of the differential equations that constitute the order-1 cycle. The section containing the impulse line and the trajectory is known as the inside of the order-1 cycle.

Definition 3 (see [9]). We assume that M and N are both straight lines and define a new number axis l on N . Suppose that N intersects with x -axis at point Q . Take the origin at point Q and define positive direction and unit length to be consistent with the coordinate y -axis, and then we obtain a number axis l . For any point $A \in l$, let $l(A) = a$ be coordinate of point A . Assume further that the trajectory through point A via k th impulsive intersects with N at point B_k , and then set $l(B_k) = b_k$, point B_k is called the order- k successor point of point A , and $F_k(A)$ is known as the order- k successor function of point A , where $F_k(A) = l(B_k) - l(A) = b_k - a$, $k = 1, 2, \dots$

Lemma 4 (see [9]). The successor function $F_k(A)$ is continuous.

Lemma 5 (see [11]). The T -periodic solution $(x, y) = (\xi(t), \eta(t))$ of the system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad \phi(x, y) \neq 0, \\ \Delta x &= \xi(x, y), \quad \Delta y = \eta(x, y), \quad \phi(x, y) = 0. \end{aligned} \quad (4)$$

is orbitally asymptotically stable if the Floquet multiplier μ satisfies the condition $|\mu| < 1$, where

$$\mu = \prod_{k=1}^n \Delta_k \exp \left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t)) \right) dt \right] \quad (5)$$

with

$$\begin{aligned} \Delta_k &= \left(P_+ \left(\frac{\partial \beta}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \beta}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x} \right) \right. \\ &\quad \left. + Q_+ \left(\frac{\partial \alpha}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) \right) \\ &\quad \times \left(P \frac{\partial \phi}{\partial x} + Q \frac{\partial \phi}{\partial y} \right)^{-1} \end{aligned} \quad (6)$$

and $P, Q, \partial\alpha/\partial x, \partial\alpha/\partial y, \partial\beta/\partial x, \partial\beta/\partial y, \partial\phi/\partial x, \partial\phi/\partial y$, which are calculated for the points $(\xi(t_k), \eta(t_k))$, $P_+ = P(\xi(t_k^+), \eta(t_k^+))$, and $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$, where $\phi(x, y)$ is a sufficiently smooth function so $\text{grad } \phi(x, y) \neq 0$, and t_k ($k \in \mathbb{N}$) is the time of the k th jump.

Lemma 6 (see [17]). Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a one-parameter family of the C^2 map that satisfies

- (i) $F(0, \mu) = 0$,
- (ii) $(\partial F/\partial x)(0, 0) = 1$,
- (iii) $(\partial^2 F/\partial x \partial \mu)(0, 0) > 0$,
- (iv) $(\partial^2 F/\partial x^2)(0, 0) < 0$.

F has two branches of fixed points for μ near zero. The first branch is $x_1(\mu) = 0$ for all μ . The second bifurcating branch $x_2(\mu)$ changes its value from negative to positive as μ increases through $\mu = 0$ with $x_2(0) = 0$. The fixed points of the first branch are stable if $\mu < 0$ and unstable if $\mu > 0$, whereas those of the bifurcating branch have the opposite stability.

Lemma 7 (see [10]). In system (1), if an order-1 periodic solution where there is no singular point is orbitally attractive, the order-1 periodic solution is orbitally asymptotically stable.

Lemma 8. In system (1), one supposes that there exists an order-1 periodic solution where the crossover points of the order-1 periodic solution for the impulsive set and phase set are points C and D , respectively, and $y_D > \text{radh}(K - (1 - p)h)(1 - p)/(aK - r(K - (1 - p)h))$. If a trajectory is attracted by the order-1 periodic solution, the order-1 periodic solution is orbitally stable.

Proof. For all $S \in Q$, $y_S \in [y_B, y_O]$, point S does not belong to the set of periodic solutions. Therefore, the combination of the order-1 and order-2 successor function of point S is one of the following:

$$\begin{aligned} [F_1(S) > 0, F_2(S) > 0], & \quad [F_1(S) > 0, F_2(S) < 0], \\ [F_1(S) < 0, F_2(S) > 0], & \quad [F_1(S) > 0, F_2(S) < 0]. \end{aligned} \quad (7)$$

Set $d_n = F_{2n-1}(S) - F_{2n}(S)$.

- (i) $F_1(S) > 0$ and $F_2(S) < 0$

If $F_1(S) > 0$ and $F_2(S) < 0$, then

$$\begin{aligned} F_1(S) &< F_3(S) < \cdots < F_{2n-1}(S), \\ F_2(S) &> F_4(S) > \cdots > F_{2n}(S), \\ F_{2n-1}(S) &> F_{2n}(S), \quad d_n > d_{n-1} > 0, \end{aligned} \quad (8)$$

where $2n \leq k$. If $n = 1, 2, 3$, it is obvious that (i) holds. Suppose that (i) holds when $n = j$. Now set $n = j + 1$. For the trajectory with the initial point order- $2j - 1$ successor point, its order-1 successor point is the order- $2j$ successor point of point S , its order-2 successor point is the order- $2j + 1$ successor point of point S , and its order-3 successor point is the order- $2j + 2$ successor point of point S . It is obvious that

$F_{2j-1} < F_{2j+1}, F_{2j} > F_{2j+2}, F_{2j+1}(S) > F_{2j+2}(S)$, and $d_{j+1} > d_j$. Therefore, (i) holds.

Similar to (i), we have

- (ii) $F_1(S) < 0$ and $F_2(S) > 0$

$$\begin{aligned} F_1(S) &> F_3(S) > \cdots > F_{2n-1}(S), \\ F_2(S) &< F_4(S) < \cdots < F_{2n}(S), \end{aligned} \quad (9)$$

$$F_{2n-1}(S) < F_{2n}(S), \quad d_n < d_{n-1} < 0,$$

- (iii) $F_1(S) < 0$ and $F_2(S) < 0$.

If $F_1(S) < 0$ and $F_2(S) < 0$, then

$$\begin{aligned} F_1(S) &< F_3(S) < \cdots < F_{2n-1}(S), \\ F_2(S) &> F_4(S) > \cdots > F_{2n}(S), \\ F_{2n}(S) &> F_{2n-1}(S), \\ 0 > d_n &= \alpha_{n-1}d_{n-1}, \quad (0 < \alpha_n < 1). \end{aligned} \quad (10)$$

If $n = 1, 2, 3$, it is obvious that (iii) holds. Suppose that (iii) holds when $n = j$. Now set $n = j + 1$. For the trajectory with the initial point order- $2j - 1$ successor point, its order-1 successor point is the order- $2j$ successor point of point S , its order-2 successor point is the order- $2j + 1$ successor point of point S , and its order-3 successor point is the order- $2j + 2$ successor point of point S . It is obvious that $F_{2j-1} < F_{2j+1}, F_{2j} > F_{2j+2}, F_{2j+1}(S) < F_{2j+2}(S)$, and $d_{j+1} < d_j$. Therefore, (iii) holds. Moreover, based on $0 > d_n = \alpha_{n-1}d_{n-1}, (0 < \alpha_n < 1)$, it is known that $d_n = \alpha_1 \cdots \alpha_{n-1}d_1$ because $0 < \alpha_n < 1$, so $\lim_{n \rightarrow +\infty} d_n = 0$.

Similar to (iii), we have

$$\begin{aligned} \text{(iv) } F_1(S) &> 0 \text{ and } F_2(S) > 0 \\ F_1(S) &> F_3(S) > \cdots > F_{2n-1}(S), \\ F_2(S) &< F_4(S) < \cdots < F_{2n}(S), \\ F_{2n-1}(S) &> F_{2n}(S), \\ 0 < d_n &= \alpha_{n-1}d_{n-1}, \quad (0 < \alpha_n < 1). \end{aligned} \quad (11)$$

Therefore, the trajectory with the initial point S is attracted by an order-1 periodic solution if case (iii) or case (iv) holds.

According to (iv), the trajectory with the initial point B is attracted by an order-1 periodic solution. Let B_k be the order- k successor point of point B . It is easy to show that the trajectory with the initial point B_2 is attracted by the order-1 periodic solution. Therefore, if we take a point U between point B and point B_2 , $F_2(U) > 0$ and $F_1(U) > 0$, while according to (iv), the trajectory with the initial point U is attracted by the order-1 periodic solution. Similarly, any trajectory with an initial point that belongs to a phase set between y_S and y_H is attracted by the order-1 periodic solution, where $y_H = \text{radh}(K - (1 - p)h)(1 - p)/(aK - r(K - (1 - p)h))$ is a crossover point of the vertical line and the phase set (see Section 3). Obviously, the order-1 periodic solution is orbitally attractive. According to Lemma 7, the order-1 periodic solution is also orbitally stable. \square

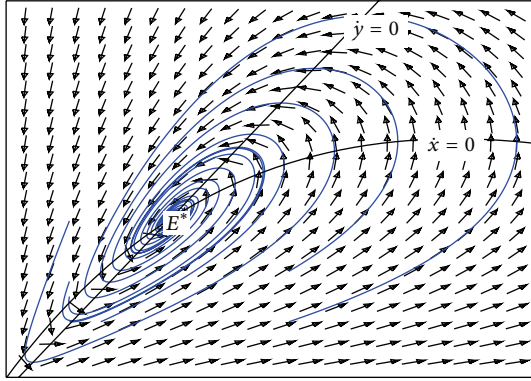


FIGURE 1: A vector graph of system (1): the black curve, $\dot{x} = 0$, denote vertical isocline, and the black curve, $\dot{y} = 0$, denotes horizontal isocline. The blue curves denote the trajectory in system (1).

Similar to Lemma 8, we have Lemma 9 as follows.

Lemma 9. *In system (1), one supposes that there exists an order-1 periodic solution where the crossover points of the order-1 periodic solution for the impulsive set and phase set are points C and D , respectively, and $y_D \leq radh(K - (1 - p)h)(1 - p)/(aK - r(K - (1 - p)h))$. If a trajectory is attracted by the order-1 periodic solution, the order-1 periodic solution is orbitally semistable at least.*

3. Main Results

First, we consider the case of system (1) without an impulsive effect. Obviously, $F = f(A) = rad(1 - A/K)A/(a - r(1 - A/K))$ is a vertical line and $F = g(A) = ((\epsilon a - adm)/m)A$ is a horizontal isocline. A direct calculation shows that $(0, K)$ is a saddle while E^* is a stable positive focus in the condition $((r - a + m)e^2 + m^2d(ad - e))^2 > 4e^2m(re - ea + adm)(e - md)$, $a < re/(e - md)$, $e > md$, $a(e^2 - m^2d^2) < (re + me - m^2d)e$, where $E^* = (A^*, F^*)$, $F^* = (a(e - md)/r)A^*$, and $A^* = K(re - ea + mad)/re$. The vector graph of system (1) is shown in Figure 1. Throughout this paper, we suppose that the condition always holds based on ecological practice, where Q and M are the impulsive set and phase set, respectively, and $h < A^*$. Next, we discuss the order-1 periodic solution of system (1).

3.1. Existence of Order-1 Periodic Solution for System (1)

3.1.1. The Case Where $\tau = 0$. In this subsection, we will derive some basic properties for the following subsystem of system (1), where fish, $F(t)$, is absent:

$$\begin{aligned} \frac{dA}{dt} &= rA \left(1 - \frac{A}{K} \right), \quad A \neq h \\ \Delta A &= -pA, \quad A = h. \end{aligned} \quad (12)$$

Setting $A_0 = A(0) = (1 - p)h$ produces the following solution of system (12): $A(t) = K(1 - p)h \exp(r(t - nT))/(K - (1 - p)h + (1 - p)h \exp(r(t - nT)))$. If we let $T = (1/r) \ln((K - (1 - p)h)/(K - (1 - p)h \exp(rT)))$, then $A(T) = h$ and $A(T^+) = (1 - p)h$. This means that system (1) has the following semitrivial periodic solution:

$$\begin{aligned} A(t) &= \frac{K(1 - p)h \exp(r(t - nT))}{K - (1 - p)h + (1 - p)h \exp(r(t - nT))}, \\ F(t) &= 0, \end{aligned} \quad (13)$$

where $t \in (nT, (n + 1)T]$, $n \in \mathbb{N}$, which is implied by $(\xi(t), 0)$. Thus, the following theorem is obtained.

Theorem 10. *There exists a semitrivial order-1 periodic solution (13) in system (1), which is orbitally asymptotically stable if*

$$-1 < q < \left(\frac{(1 - p)(K - h)}{K - (1 - p)h} \right)^{(\epsilon - md)/rd} - 1. \quad (14)$$

Proof. It is known that $P(A, F) = rA(1 - A/K) - aAF/(F + adA)$, $Q(A, F) = \epsilon aAF/(F + adA) - mF$, $\alpha(A, F) = -pA$, $\beta(A, F) = qF$, $\phi(A, F) = A - h$, $(\xi(T), \eta(T)) = (h, 0)$, and $(\xi(T^+), \eta(T^+)) = ((1 - p)h, 0)$. Using Lemma 5 and a straightforward calculation, it is possible to obtain

$$\begin{aligned} \frac{\partial P}{\partial A} &= r \left(1 - \frac{2}{K}A \right) - \frac{aF^2}{(F + adA)^2}, \\ \frac{\partial Q}{\partial F} &= \frac{\epsilon da^2 A^2}{(F + adA)^2} - m, \\ \frac{\partial \alpha}{\partial A} &= -p, \quad \frac{\partial \alpha}{\partial F} = 0, \\ \frac{\partial \beta}{\partial A} &= 0, \quad \frac{\partial \beta}{\partial F} = q, \\ \frac{\partial \phi}{\partial A} &= 1, \quad \frac{\partial \phi}{\partial F} = 0, \\ \Delta_1 &= \left(P_+ \left(\frac{\partial \beta}{\partial F} \frac{\partial \phi}{\partial A} - \frac{\partial \beta}{\partial A} \frac{\partial \phi}{\partial F} + \frac{\partial \phi}{\partial A} \right) \right. \\ &\quad \left. + Q_+ \left(\frac{\partial \alpha}{\partial A} \frac{\partial \phi}{\partial F} - \frac{\partial \alpha}{\partial F} \frac{\partial \phi}{\partial A} + \frac{\partial \phi}{\partial F} \right) \right) \\ &\quad \times \left(P \frac{\partial \phi}{\partial A} + Q \frac{\partial \phi}{\partial F} \right)^{-1} \\ &= \frac{P^+(\xi(T^+), \eta(T^+))(1 + q)}{P(\xi(T), \eta(T))} \\ &= (1 - p)(1 + q) \frac{K - (1 - p)h}{K - h}. \end{aligned} \quad (15)$$

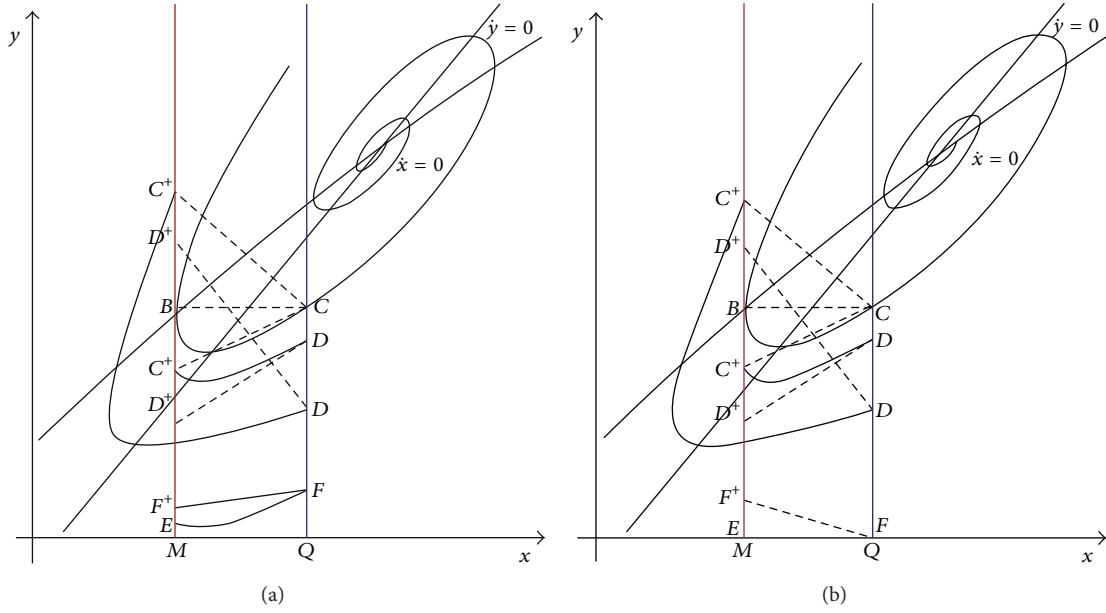


FIGURE 2: (a) is the proof on Theorem 12; (b) is the proof on Theorem 13. In (a) and (b), the red line M and the blue line Q represent impulsive set and phase set, respectively. The black curve, $\dot{x} = 0$, denotes vertical isocline, and the black curve, $\dot{y} = 0$, denotes horizontal isocline.

Furthermore,

$$\begin{aligned} & \exp \left[\int_0^T \left(\frac{\partial P}{\partial A} (\xi(t), \eta(t)) + \frac{\partial Q}{\partial F} (\xi(t), \eta(t)) \right) dt \right] \\ &= \exp \left[\int_0^T \left(r \left(1 - \frac{2}{K} \xi(t) \right) + \frac{\varepsilon}{d} - m \right) dt \right] \\ &= \left(\frac{K - (1-p)h}{(K-h)(1-p)} \right)^{1+((\varepsilon-md)/rd)} \left(\frac{K - (1-p)h}{K-h} \right)^{-2}. \end{aligned} \quad (16)$$

Therefore, it is possible to obtain the Floquet multiplier μ by direct calculation as follows:

$$\begin{aligned} \mu &= \prod_{k=1}^n \Delta_k \exp \left[\int_0^T \left(\frac{\partial P}{\partial x} (\xi(t), \eta(t)) \right. \right. \\ &\quad \left. \left. + \frac{\partial Q}{\partial y} (\xi(t), \eta(t)) \right) dt \right] \quad (17) \\ &= (1+q) \left(\frac{K - (1-p)h}{(1-p)(K-h)} \right)^{(\varepsilon-md)/rd}. \end{aligned}$$

Thus, $|\mu| < 1$ if (14) holds. This completes the proof. \square

Remark 11. If $q^* = ((1-p)(K-h)/(K-(1-p)h))^{(\varepsilon-md)/rd} - 1$, a bifurcation may occur if $q = q^*$ for $|\mu| = 1$, whereas a positive periodic solution may emerge if $q > q^*$.

Theorem 12. *There exists a positive order-1 periodic solution in system (1) if $q > q^*$ where the semitrivial periodic solution is orbitally unstable.*

Proof. Because $h < A^*$, M and Q are both in the left E^* . The trajectory that passes through point B tangents to M at

point B and intersects with Q at point C . Thus, there may be three cases of phase point (C^+) for point C as follows (see Figure 2(a)).

Case I ($y_B = y_{C^+}$). In this case, it is obvious that BCB is an order-1 periodic solution.

Case II ($y_B < y_{C^+}$). Point C^+ is the order-1 successor point of point B , so the order-1 successor function of point B is greater than zero; that is, $F_1(B) = y_{C^+} - y_B > 0$. In addition, the trajectory with the initial point C^+ intersects with the set of impulses Q at point D and reaches D^+ via the impulsive effect. Due to the disjointedness of the different trajectories, it is easy to see that point D^+ is located below point C^+ . Therefore, the successor function $F_1(C^+) < 0$. According to Lemma 4, a point $E \in M$ is known to exist such that $F_1(E) = 0$, so there exists an order-1 periodic solution for system (1).

Case III ($y_B > y_{C^+}$). According to $y_B > y_{C^+}$, the order-1 successor point of point B is located below point B , so $F_1(B) < 0$. If we suppose that p_0 is a crossover point of the semitrivial periodic solution and impulsive set, because the semitrivial periodic solution is orbitally unstable, then there exists a point $E \in \cup(p_0, \delta)$ such that $F_1(E) \geq 0$. If $F_1(E) < 0$, the trajectory with the initial point E is attracted by the semitrivial periodic solution and, according to Lemma 9, the semi-periodic solution is orbitally stable. Obviously, this is a contradiction, so $F_1(E) \geq 0$. Thus, there exists a order-1 positive periodic solution when $F_1(E) = 0$. According to Lemma 4, a point $K \in M$ is known to exist such that $F_1(K) = 0$ when $F_1(E) > 0$. Therefore, there exists an order-1 periodic solution for system (1).

The proof is completed. \square

3.1.2. *The Case Where $\tau > 0$.* In this case, we suppose that $h < A^*$ and the following theorem is described.

Theorem 13. *There exists a positive order-1 periodic solution for system (1) if $\tau > 0$ and $h < A^*$.*

Proof (see Figure 2(b)). The method for this proof is similar to the method for Theorem 12. The main difference is the proof for the case $y_B > y_{C^+}$. Suppose that E is a crossover point for a semitrivial periodic solution and an impulsive set. The trajectory with initial point E intersects the impulsive set at F . Obviously, $y_E = y_F = 0$. Because $\tau > 0$, $y_{F^+} = (1+q)y_F + \tau > 0 = y_F$. Thus, there exists a positive order-1 periodic solution for system (1), which completes the proof. \square

In summary, system (1) has a stable semitrivial periodic solution or a positive order-1 periodic solution when $\tau \geq 0$. Furthermore, using the analogue of the Poincaré criterion, the stability of positive order-1 periodic solution is obtained.

Theorem 14. *For any $\tau > 0$, $q > -1$, or $\tau = 0$, $q \geq q^*$, the order-1 periodic solution of system (1) is orbitally stable if the following condition holds:*

$$\left| (1-p)(1+q) \times \left(r \left(1 - \frac{(1-p)h}{K} \right) - \frac{a((1+q)\eta_0 + \tau)}{((1+q)\eta_0 + \tau) + ad(1-p)h} \right) \times \left(r \left(1 - \frac{h}{K} \right) - \frac{a\eta_0}{\eta_0 + adh} \right)^{-1} \times \exp \left(\int_0^T G(t) dt \right) \right| < 1, \quad (18)$$

where $G(t) = (\partial P / \partial A)(\xi(t), \eta(t)) + (\partial Q / \partial F)(\xi(t), \eta(t))$.

Proof. We suppose that the period of the order-1 periodic solution is T , so the order-1 periodic solution intersects the impulsive set at $E(h, \eta_0)$ and phase set at $E^+((1-p)h, (1+q)\eta_0 + \tau)$. Let $(\xi(t), \eta(t))$ be the expression of the order-1 periodic solution. The difference between this case and the case in Theorem 10 is that $(\xi(T), \eta(T)) = (h, \eta_0)$, $(\xi(T^+), \eta(T^+)) = ((1-p)h, (1+q)\eta_0 + \tau)$, whereas the others are the same. Thus, we have

$$\Delta_1 = (1-p)(1+q) \times \left(r \left(1 - \frac{(1-p)h}{K} \right) - \frac{a((1+q)\eta_0 + \tau)}{((1+q)\eta_0 + \tau) + ad(1-p)h} \right)$$

$$\times \left(r \left(1 - \frac{h}{K} \right) - \frac{a\eta_0}{\eta_0 + adh} \right)^{-1},$$

$$\mu_2 = \Delta_1 \exp \left(\int_0^T G(t) dt \right). \quad (19)$$

According to condition (18), $|\mu_2| < 1$, so the order-1 periodic solution is orbitally stable using the analogue of the Poincaré criterion. The proof is complete. \square

3.2. Bifurcation and the Movement of the Order-1 Periodic Solution

3.2.1. *Transcritical Bifurcation.* In this subsection, we will discuss the bifurcation near the semitrivial periodic solution. The following Poincaré map P is used:

$$y_k^+ = (1+q)g(y_{k-1}^+), \quad (20)$$

where we choose section $S_0 = (1-p)h$ as a Poincaré section. If we set $0 \leq u = y_k^+$ at a sufficiently small value, the map can be written as follows:

$$u \mapsto (1+q)g(u) \equiv G(u, q). \quad (21)$$

Using Lemma 6, the following theorem can be obtained.

Theorem 15. *A transcritical bifurcation occurs when $q = q^*$, $\tau = 0$. Therefore, a stable positive fixed point appears when the parameter q changes through q^* from left to right. Correspondingly, system (1) has a stable positive periodic solution if $q \in (q^*, q^* + \delta)$ with $\delta > 0$.*

Proof. The values of $g'(u)$ and $g''(u)$ must be calculated at $u = 0$ where $0 \leq u \leq u_0$. Here, $u_0 = radKh(1-p)(K - (1-p)h)/(aK - r(K - (1-p)h))$. Thus, system (1) can be transformed as follows:

$$\frac{dF}{dA} = \frac{Q(A, F)}{P(A, F)}, \quad (22)$$

where $P(A, F) = rA(1 - A/K) - aAF/(F + adA)$, $Q(A, F) = \varepsilon aAF/(F + adA) - mF$.

Let $(A, F(A; A_0, F_0))$ be an orbit of system (22) and $A_0 = (1-p)h$, $F_0 = u$, $0 \leq u \leq u_0$. Then,

$$F(A; (1-p)h, u) \equiv F(A, u), \quad (1-p)h \leq A \leq h, \quad 0 \leq u \leq u_0. \quad (23)$$

Using (23),

$$\begin{aligned} \frac{\partial F(A, u)}{\partial u} &= \exp \left[\int_{(1-p)h}^A \frac{\partial}{\partial F} \left(\frac{Q(s, F(s, u))}{P(s, F(s, u))} \right) ds \right], \\ \frac{\partial^2 F(A, u)}{\partial u^2} &= \frac{\partial F(A, u)}{\partial u} \\ &\times \int_{(1-p)h}^A \frac{\partial^2}{\partial F^2} \left(\frac{Q(s, F(s, u))}{P(s, F(s, u))} \right) \frac{\partial F(s, u)}{\partial u} ds, \end{aligned} \quad (24)$$

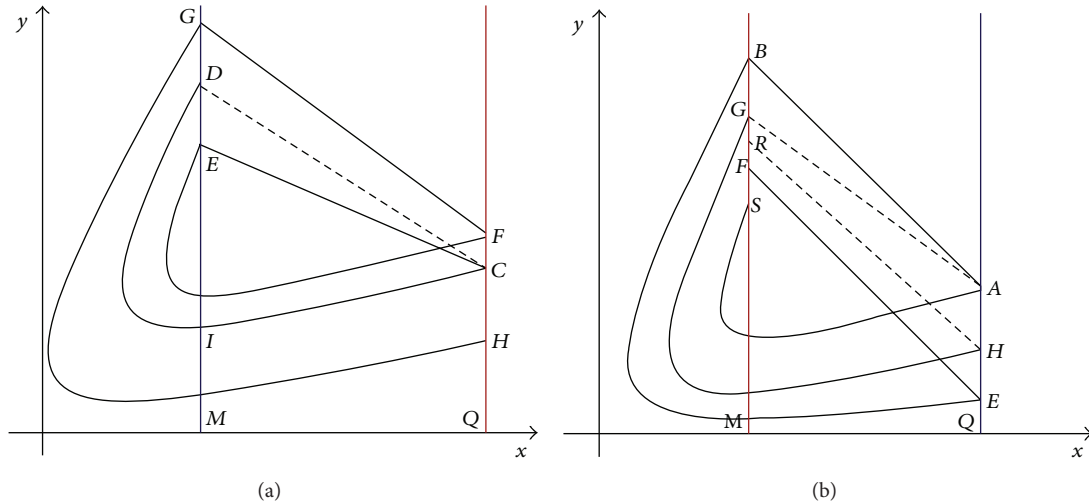


FIGURE 3: It represents the proof on Theorem 17. In (a) and (b), the line M and the line Q represent impulsive set and phase set, respectively. The curves represent trajectory of system (1), and the lines denote impulsive line of system (1).

and it can clearly be deduced that $\partial F(A, u)/\partial u > 0$, and

$$\begin{aligned} g'(0) &= \frac{\partial F(h, 0)}{\partial u} \\ &= \exp \left(\int_{(1-p)h}^h \frac{\partial}{\partial F} \left(\frac{Q(s, F(s, 0))}{P(s, F(s, 0))} \right) ds \right) \\ &= \exp \left(\int_{(1-p)h}^h \frac{K(\varepsilon - dm)}{rs(K - s)} ds \right) \\ &= \left(\frac{K - (1-p)h}{(1-p)(K - h)} \right)^{(\varepsilon - dm)/rd}. \end{aligned} \quad (25)$$

Furthermore,

$$g''(0) = g'(0) \int_{(1-p)h}^h m(s) \frac{\partial F(s, 0)}{\partial u} ds, \quad (26)$$

where $m(s) = (\partial^2/\partial F^2)(Q(s, F(s, 0))/P(s, F(s, 0))) = 2K(\varepsilon rs - K(\varepsilon r + adm - \varepsilon a))/ad^2r^2s^2(K - s)^2$, $s \in [(1-p)h, h]$. Because u is sufficiently small, this yields $\varepsilon rs - K(\varepsilon r + adm - \varepsilon a) < 0$. It can be determined that $m(s) < 0$, $s \in [(1-p)h, h]$. Therefore,

$$g''(0) < 0. \quad (27)$$

The next step is to check whether the following conditions are satisfied.

- (a) It is easy to see that $G(0, q) = 0$, $q \in (0, \infty)$.
- (b) Using (25), $\partial G(0, q)/\partial u = (1 + q)g'(0) = (1 + q)((K - (1-p)h)/(1-p)(K - h))^{(\varepsilon - dm)/rd}$, which yields $\partial G(0, q^*)/\partial u = 1$. This means that $(0, q^*)$ is a fixed point with an eigenvalue of 1 in map (20).
- (c) Because (25) holds, $\partial^2 G(0, q^*)/\partial u \partial q = g'(0) > 0$.
- (d) Finally, inequality (27) implies that $\partial^2 G(0, q^*)/\partial u^2 = (1 + q^*)g''(0) < 0$.

These conditions satisfy the conditions of Lemma 6. This completes the proof. \square

3.2.2. Movement of the Order-1 Periodic Solution. In this subsection, we will discuss the movement of the order-1 periodic solution with variable parameters. The following theorem is required.

Theorem 16. *The rotation direction of the pulse line is clockwise if q changes from $q = 0$ to $q > 0$.*

Proof. Let θ be the angle of the pulse line and the x -axis. Then, $\tan \theta = \Delta F/\Delta A = Q/P$, so $\theta = \tan^{-1}(Q/P)$. Furthermore, $\partial \theta/\partial q = (1/(P^2 + Q^2)) \left| \frac{\partial P}{\partial q} \frac{Q}{P} - \frac{\partial Q}{\partial q} \right| = (1/(P^2 + Q^2))(-pAF) < 0$. Therefore, θ is a monotonically decreasing function of q . This completes the proof. \square

The existence of an order-1 periodic solution was proved in the previous analysis, so we assume that there exists an order-1 periodic solution when $q = q^*$ and $\tau > 0$, where the crossover points of the order-1 periodic solution for the impulsive set and the phase set are points C and D, respectively. The following theorem is then described.

Theorem 17. *In system (1), one supposes that there exists a stable and positive order-1 periodic solution if $q = q^*$, $\tau > 0$, and $y_D > radh(K - (1-p)h)(1-p)/(aK - r(K - (1-p)h))$. The order-1 periodic solution moves toward the inside of the order-1 periodic solution along the pulse set and it is orbitally stable when q changes appropriately from $q = q^*$ to $q < q^*$.*

Proof (see Figure 3(a)). The order-1 periodic solution breaks when q changes. According to Theorem 16, point E, which is the phase point of point C, is located below point D when q^* decreases. Because $y^+ = y + qy + \tau$ (here $y > 0$) is a monotonically increasing and continuing function of q , there exists $\varepsilon > 0$ such that $y_D < y_E < y_I$. Figure 3(a) shows that point E is the order-1 successor point of point D, while point G is the order-1 successor point of point E, so $F_1(D) < 0$, $F_1(E) > 0$. Therefore, there exists a point K between point D and E such that $F_1(K) = 0$. According to the disjointedness

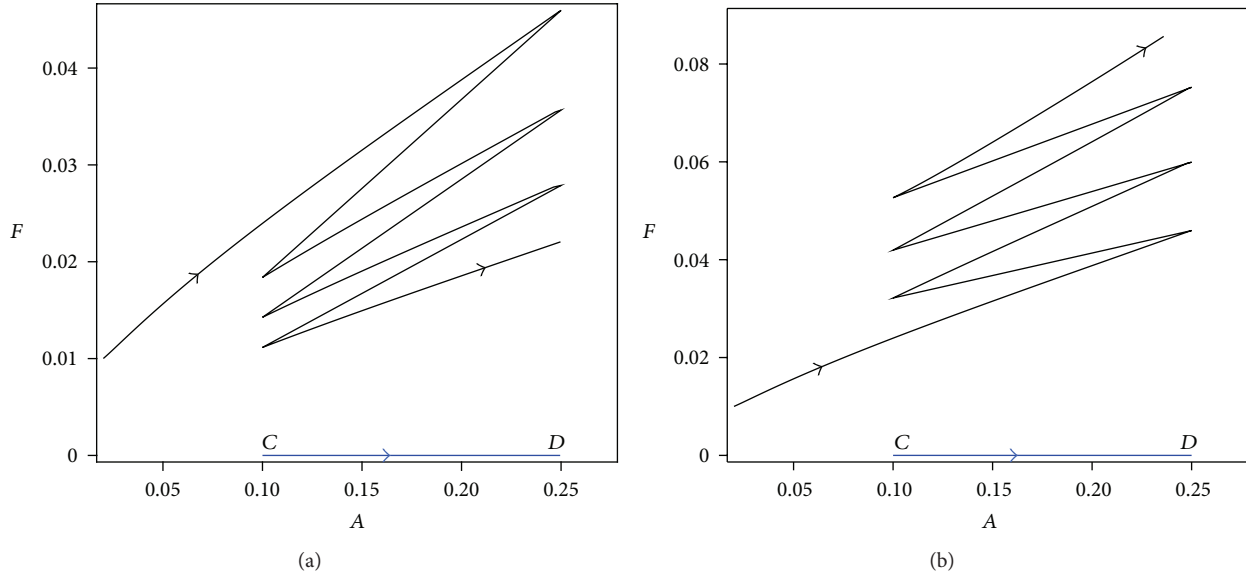


FIGURE 4: Trajectories with the initial point (0.02, 0.01) in system (1) where (a) $q = -0.6$ and (b) $q = -0.3$, where the blue line CD displays the semitrivial of system (1).

of the different trajectories, the order-1 periodic solution is inside the order-1 periodic solution $DICD$.

Next, the orbital stability can be established based on the following proof (see Figure 3(b)).

The order-1 periodic solution $DICD$ is orbitally stable, so according to Lemma 8 and the disjointedness of the pulse line, there exists a point S between points D and I (see Figure 3(a)) such that $F_1(S) > 0$, $F_2(S) > 0$. We suppose that the reduction in q^* is $\varepsilon > 0$.

If $\varepsilon = 0$, point B is the order-1 successor point of S and point F is the order-2 successor point of point S . Because of $y_B = (1 + q^*)y_A + \tau$, $y_F = (1 + q^*)y_E + \tau$, so $F_1(S) = y_B - y_S = (1 + q^*)y_A + \tau - y_S > 0$, $F_2(S) = y_F - y_S = (1 + q^*)y_E + \tau - y_S > 0$.

While $\varepsilon > 0$, the order-1 and order-2 successor points of point S are points G and R , respectively, where $y_G = (1 + q^* - \varepsilon)y_A + \tau$, $y_R = (1 + q^* - \varepsilon)y_H + \tau$. Therefore, $F_1(S) = y_G - y_S = (1 + q^* - \varepsilon)y_A + \tau - y_S$, $F_2(S) = y_R - y_S = (1 + q^* - \varepsilon)y_H + \tau - y_S$, where set $F_1(S) = F_1^1(S)$ and $F_2(S) = F_2^2(S)$ distinguish the successor function between $\varepsilon = 0$ and $\varepsilon > 0$. Therefore, we have the following: $F_1^1(S) = (1 + q^* - \varepsilon)y_A + \tau - y_S = (1 + q^*)y_A + \tau - y_S - \varepsilon y_A$. Because $(1 + q^*)y_A + \tau - y_S > 0$, so $F_1^1(S) > 0$ when $0 < \varepsilon < ((1 + q^*)y_A + \tau - y_S)/y_A \xrightarrow{\text{define}} \varepsilon_1$.

In addition,

$$\begin{aligned} F_2(S) - F_2^2(S) &= (1 + q^*)y_E + \tau - y_S \\ &\quad - (1 + q^* - \varepsilon)y_H + \tau - y_S \\ &= (1 + q^*)(y_E - y_H) + \varepsilon y_H. \end{aligned} \quad (28)$$

Obviously, $F_2(S) - F_2^2(S) < 0$ when $0 < \varepsilon < ((1 + q^*)(y_H - y_E)/y_H) \xrightarrow{\text{define}} \varepsilon_2$, where $y_H > y_E$ from Figure 3(b). If we set $\varepsilon^* = \min(\varepsilon_1, \varepsilon_2)$, $F_1^1(S) > 0$ and $F_2^2(S) > 0$ when $\varepsilon \in (0, \varepsilon^*)$. From case (iv) in Lemma 8, the trajectory with an initial point S is attracted by the periodic solution $DICD$.

According to Lemma 8, the order-1 periodic solution $DICD$ is orbitally stable. This completes the proof. \square

Similar to the method used for the proof of Theorem 17, the following theorem exists.

Theorem 18. In system (1), one supposes that there exists a stable and positive order-1 periodic solution if $q = q^*$, $\tau > 0$, and $y_D \leq \text{radh}(K - (1 - p)h)(1 - p)/(aK - r(K - (1 - p)h))$. Therefore, the order-1 periodic solution moves toward the outside of the order-1 periodic solution along the pulse set and it is orbitally stable when q changes appropriately from $q = q^*$ to $q < q^*$.

4. Numerical Results

The following numerical results are provided to illustrate the feasibility of the theoretical results. In this section, the parameters are fixed as follows: $r = 0.6$, $K = 2$, $a = 1$, $d = 0.6$, $\varepsilon = 0.5$, and $m = 0.4$. The stable positive focus is $E^* = (0.31, 0.2015)$, so $h < 0.31$.

4.1. Stability of the Semitrivial Solution. Based on the previous analysis, there exists a semitrivial solution when $\tau = 0$ in system (1). If we set $p = 0.6$ and $h = 0.25$, the semitrivial solution is $A(t) = 2 \exp(0.6154(t - nT))/(19 + \exp(0.6154(t - nT)))$ and $F(t) = 0$ with $T = 1.622609349$, where $t \in (nT, (n+1)T]$, $n \in \mathbb{N}$. Based on Remark 11, $q^* = -0.5049669337$. According to Theorem 10, the semitrivial periodic solution \overline{CDC} is orbitally stable when $q \in (-1, q^*)$, as shown in Figure 4(a) where $q = -0.6$. While $q > q^*$, the semitrivial periodic solution \overline{CDC} is unstable, as shown in Figure 4(b) where $q = -0.3$.

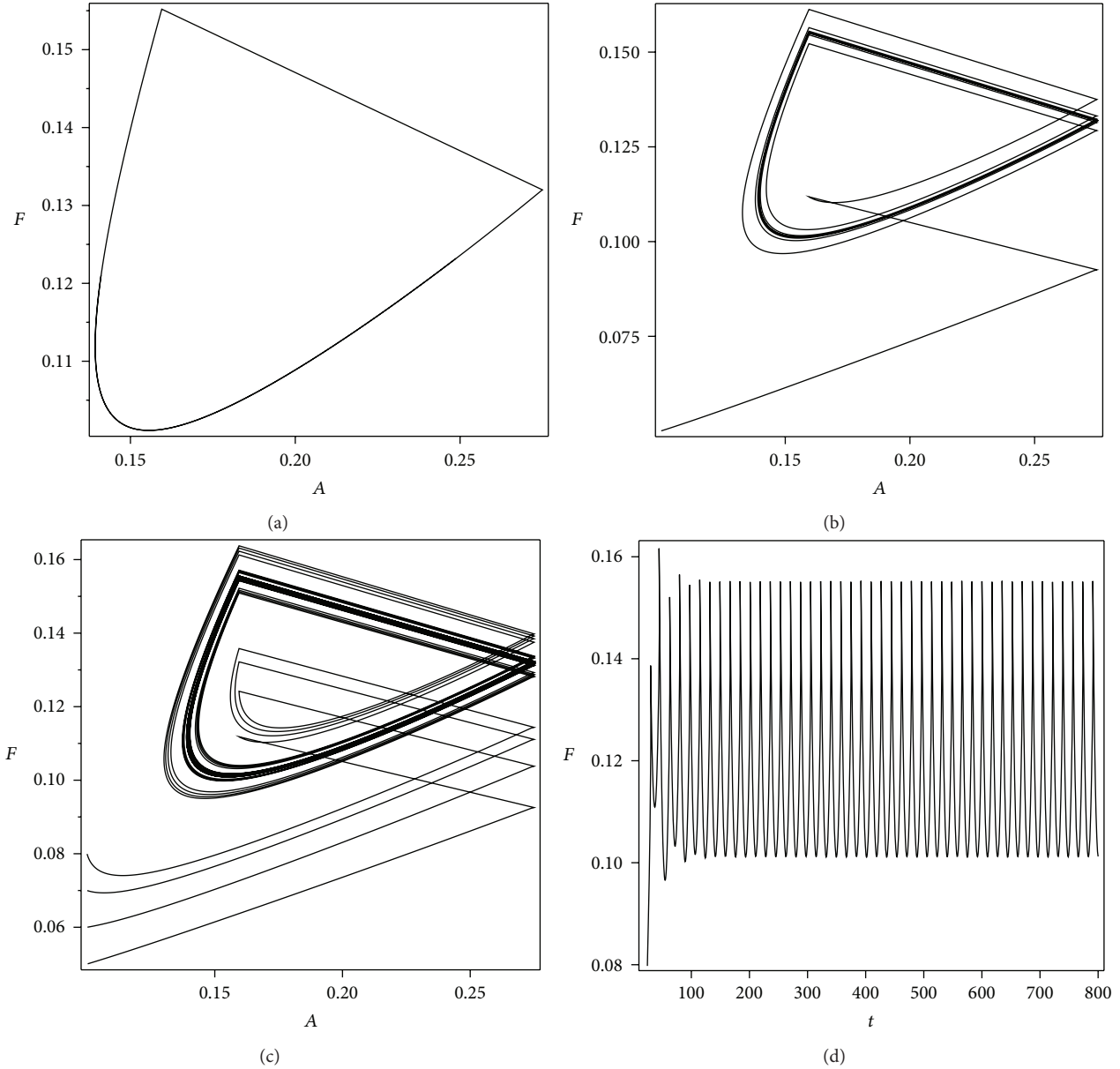


FIGURE 5: (a) an order-1 periodic solution of system (1); (b) a trajectory with the initial point $(0.02, 0.01)$ in system (1), and the trajectory is attracted by an order-1 periodic solution; (c) several trajectories with different initial point in system (1), and these trajectories are attracted by the same order-1 periodic solution; (d) the time series of system (1) corresponding to (b).

4.2. The Existence and Stability of the Order-1 Periodic Solution. According to Theorems 12 and 13, there exists a positive order-1 periodic solution for system (1). In addition, the order-1 periodic solution is orbitally asymptotically stable when the conditions of Theorem 14 or Lemma 8 hold. If we set $p = 0.6$, $h = 0.275$, $\tau = 0.01$, and $q = 0.1$ in system (1), an order-1 periodic solution exists for Figure 5(a). Furthermore, the trajectory is attracted by the order-1 periodic solution in Figure 5(b). Figures 5(c) and 5(d) prove that the order-1 periodic solution is orbitally asymptotically stable; that is, Lemma 8 is correct.

Figure 6 is provided to further consider the existence of an order-1 periodic solution of system (1). Figure 6 shows the existing regions of an order-1 periodic solution, which is the

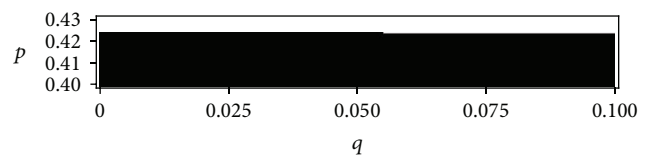


FIGURE 6: Existence of an order-1 periodic solution for system (1) versus the parameters p and q . In the black zone, there always exists an order-1 periodic solution in system (1). Note: this figure does not mean that there is no order-1 periodic solution in system (1).

part of the bifurcation of the positive stable order-1 periodic solution of system (1), where p and q are parameters.

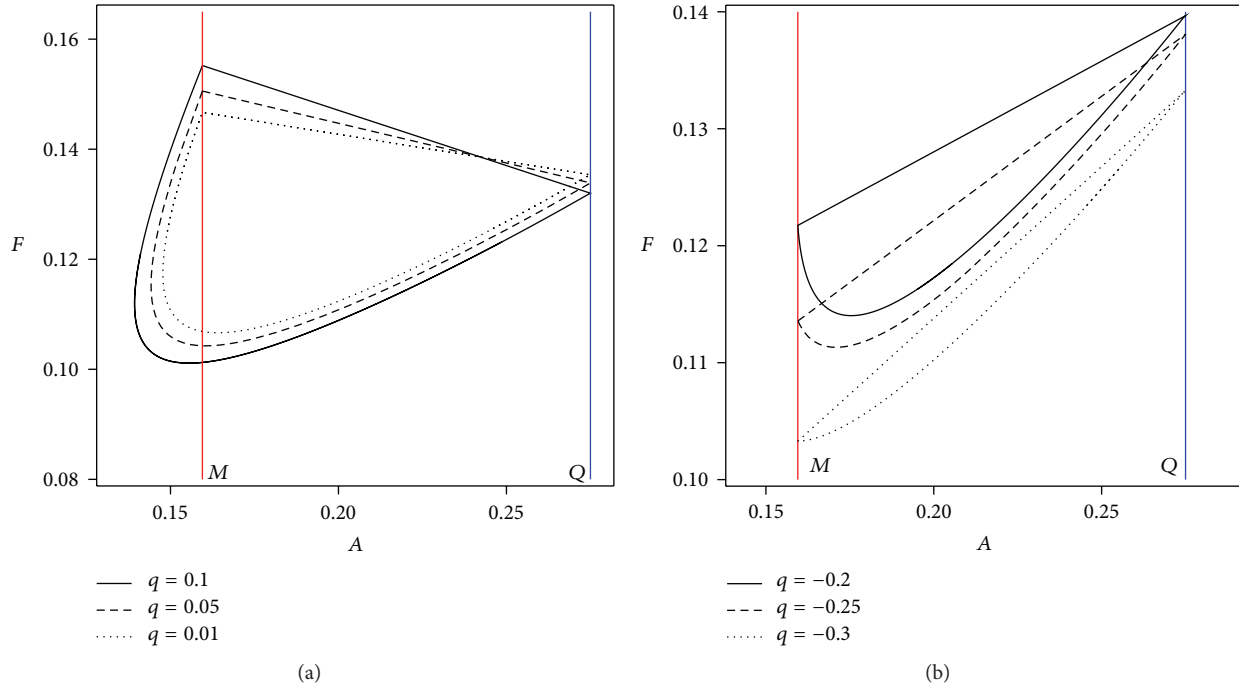


FIGURE 7: Transition of the order-1 periodic solution of system (1). The line M and the line Q represent impulsive set and phase set, respectively. Black curves show the order-1 periodic of system (1).

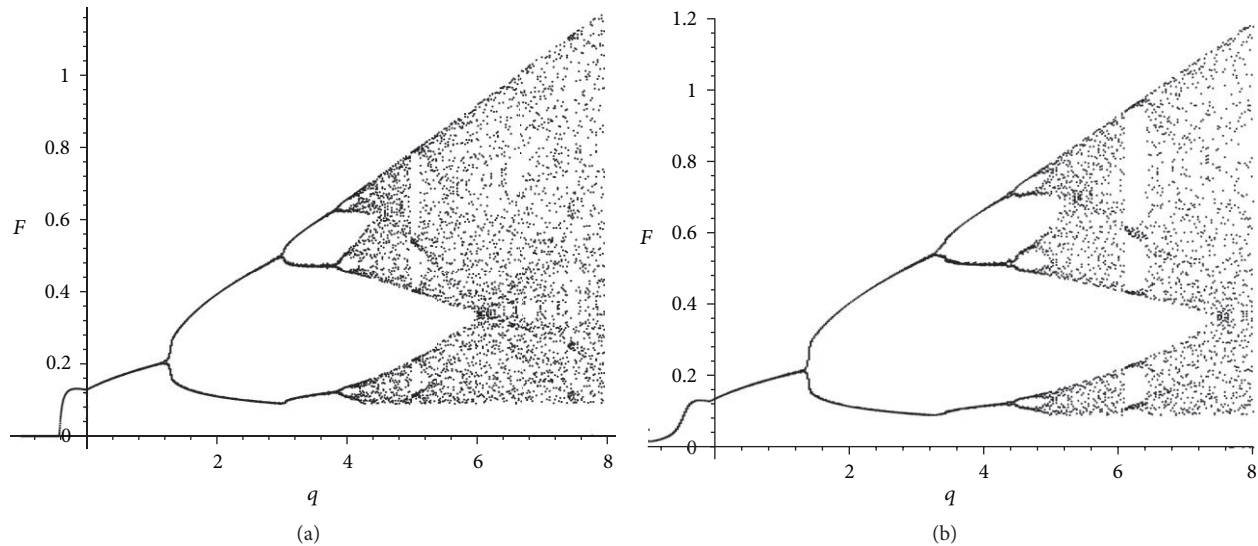


FIGURE 8: Bifurcation diagrams for system (1) $q \in (-1, 8]$, where (a) $\tau = 0$ and (b) $\tau = 0.009$.

4.3. Movement of the Order-1 Periodic Solution. From Theorems 17 and 18, the order-1 periodic solution moves toward the inside or outside of the order-1 periodic solution along the pulse set and phase set if q changes appropriately from $q = q^*$ to $q < q^*$. In Section 4.2, there exists an order-1 periodic solution when $q = 0.1$. Next, we reduce $q = 0.1$ to $q = 0.05$ and $q = 0.01$. It is then easy to see that the order-1 periodic solution moves toward the inside along the impulsive set and the phase set from Figure 7(a), while Figure 7(b) proves that an order-1 periodic solution moves toward the outside along

the impulsive set and phase set under the conditions stated in Theorem 18.

4.4. Bifurcation Analysis. To study the dynamics of system (1), a bifurcation is obtained that provides a summary of the essential dynamical behavior of system (1). The bifurcation diagrams of system (1) are plotted as a function of the bifurcation parameter q and shown in Figure 8. Due to the similarity between Figures 8(a) and 8(b), which is a flip

bifurcation of Figure 8(a), only Figure 8(a) is analyzed in detail, where $p = 0.5$, $h = 0.275$, and $\tau = 0$ in Figure 8(a) and $\tau = 0.009$ in Figure 8(b). It is obvious that the semitrivial periodic solution is stable for $q \in (-1, -0.418)$ and unstable for $q > -0.418$.

According to Theorem 15, a transcritical bifurcation occurs when $q = q^* \approx -0.418$, which leads to a positive order-1 periodic solution from a semitrivial periodic solution. As q increases, order-1 periodic solution \rightarrow order-2 periodic solution \rightarrow order-4 periodic solution, and a cascade of period-halving bifurcations leads to chaos.

5. Conclusion and Discussion

In this paper, we developed an algae-fish semicontinuous model, which we studied analytically and numerically. Theoretical mathematical studies have investigated the existence and stability of a semi-trivial periodic solution and an order-1 periodic solution of system (1), proving that the positive periodic solution emerges from the semitrivial periodic solution via a transcritical bifurcation using bifurcation theory.

In the semicontinuous system, the movement of the order-1 periodic solution was first studied theoretically, which will be useful for studying the control of algae. In system (1), the impulsive effect demonstrated the biological and chemical control of algae. Using this theory, we can study the effects of biological control on system (1). Furthermore, it will be helpful for studying the effect of increased biological and decreased chemical controls on system (1), because it is harmful to use chemical controls in this environment.

In addition, our results are useful for others systems. For example, some applications refer to the mathematical model proposed in the literature [18]. In the literature [18], Bianca and Pennisi develop a model, which is the first mathematical model that reproduces the SimTriplex results on the triplex vaccine. The model is more valuable, which takes into account both the humoral and cellular branches of the immune response and includes many realistic factors. From their work, we think that it is feasible to investigate vaccine models using our results.

Acknowledgments

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